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Where Is the "Thousand-Dollar Ko"?

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Abstract. This paper features a problem that was composed to illustrate the power of combinatorial game theory applied to Go endgame positions.

The problem is the sum of many subproblems, over a dozen of which have temperatures significantly greater than one. One of the subproblems is a conspicuous four-point ko, and there are several overlaps among other subproblems. Even though the theory of such positions is far from complete, the paper demonstrates that enough mathematics is now known to obtain provably correct, counterintuitive, solutions to some very difficult Go endgame problems.

1. Introduction

Consider the boards in Figures 1 and 2, which differ only by one stone (the White stone a knight's move south-southwest of E in Figure 1 is moved to a knight's move east-southeast of R in Figure 2). In each case, it is White's turn to move. Play will proceed according to the Ing rules, as recommended by the American Go Association in February 1994. There is no komi. Both sides have the same number of captives when the problem begins.

We will actually solve four separate problems. In Figure 1, Problem 1 is obtained by removing the two stones marked with triangles, and Problem 2 is as shown. Does the removal of the two stones matter? In Figure 2, Problem 3 is obtained by removing the two marked stones, and Problem 4 is as shown. (Problem 3 appeared on the inside cover of *Go World*, issue 70.)

In each case, assume that the winner collects a prize of $1,000. If he wins by more than one point, he can keep the entire amount, but if he wins by only one point, he is required to pay the loser $1 per move. How long will each game last?

2. Summary of the Solutions

We think that White wins Problem 1 by one point, but Black wins Problem 2 by one point.

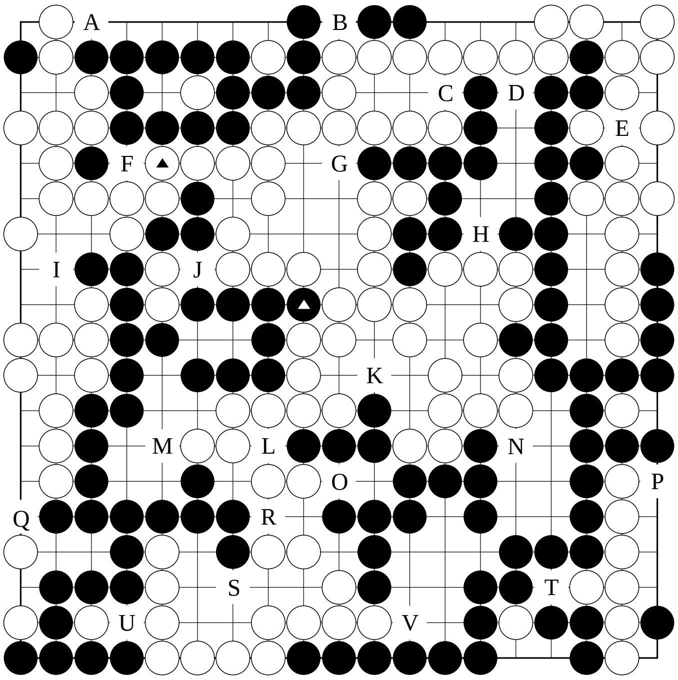


Figure 1. Starting position of Problems 1 and 2.

Despite an enormous number of possible lines of play, the assertions that suggest that conclusion are all proved mathematically, without computers. Surprisingly little read-ahead or searching is required.

For the first 53 moves, the canonical play of both problems is identical. At move 7, White must play a nominally smaller move in order to reduce the number of Black kothreats. Black begins the ko at the point marked E on move 12. But then, even though White has enough kothreats to win the ko, he must decline to fight it! Black fills the ko on move 14.

The ko located near the point marked E in Figure 1 is the only ko that is clearly visible in the problem statements shown in Figure 1. It became mistakenly known as the Thousand-Dollar Ko. It was difficult enough to stump all of the contestants in a recent contest sponsored by Ishi Press, International.

After all moves whose unchilled mathematical temperature exceeds one point (or equivalently, "two points in gote") have been played, there is an interesting and nontrivial contest on the regions that chill to infinitesimals. Using methods described in *Mathematical Go* [Berlekamp and Wolfe 1994], one shows that White has a straightforward win in Japanese scoring. However, under the Ing rules, one of the regions can assume a slightly different value because of a potential small Chinese ko. Black must play under the assumption that he can win this ko. This ko, which is difficult to foresee, is the real Thousand-Dollar Ko. It begins to affect the play soon after the first 20 moves of either problem.

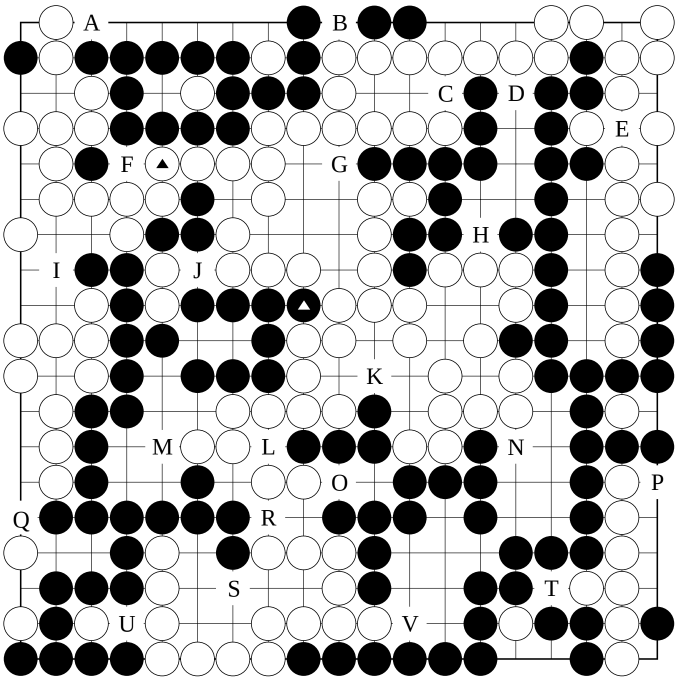


Figure 2. Starting position of Problems 3 and 4.

The players next play the regions that chill to numbers (or, in Go jargon, moves worth less than two points in gote). Because each dame can play a decisive role in the forthcoming Chinese kofight, this subsequence of play requires considerable care.

Under Japanese scoring, the game ends at move 53 with a one-point win for White. This is true in every version of the original problem.

Move 54 marks the beginning of final preparations for the decisive thousand dollar kofight, which is *not* located near E. Beginning at move 56, optimal play of Problems 1 and 2 diverge significantly. In Problem 2, Black attacks White's kothreats and wins most quickly by removing as many of them as she can before the kofight begins. But in the Problem 1, which Black cannot win, she prolongs the game by creating more kothreats of her own. White can then win only by starting the ko surprisingly early, while many nominally "removable" kothreats still remain on the board.

3. Mathematical Modeling

The philosophy that we follow is common in applied mathematics. We begin by making various assumptions that simplify the problem to one that is more easily modelled mathematically. Then we solve the mathematical problem, using whatever simplifying approximations and assumptions seem reasonable and convenient. Finally, after we have obtained the solution to the idealized problem, we verify that this solution is sufficiently robust to remain valid when the simplifications that were made to get there are all removed. The final process of verification of robustness can also be viewed as "picking away the nits".

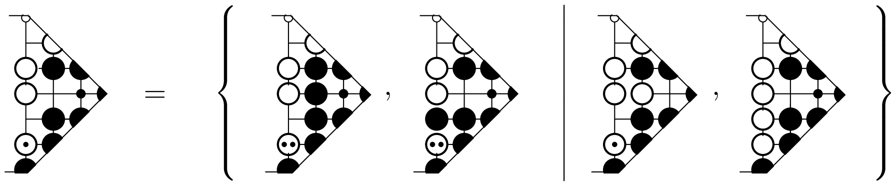
4. Chilled Values

The mathematical solution of our two problems begins by determining the chilled value of each nontrivial position in each region of the board, appropriately marked, subject to the assumption that all unmarked stones sufficiently far away from the relevant letter are "safely alive" or "immortal". These chilled values are shown in Table 1.

Most of these values can be looked up in Appendix E of *Mathematical Go* (or, perhaps more easily, in the equivalent directory on pages 71–76 of the same reference).

Many of these values include specific infinitesimals, whose notations and properties are described in *Mathematical Go.* Occasionally, when the precise value of some infinitesimal is unnecessarily cumbersome, we simply call it "ish", for "infinitesimally shifted". This now-common abbreviation was introduced long ago by John Conway, the discoverer of the theory of partizan games.

Values must also be computed for those regions that do not appear verbatim in the book. Such calculations are done as described in Chapter 2 of *Mathematical Go.* As an example of such a calculation, we consider the region Q. We have $\mathrm{Q} = \{\mathrm{Q}^L \mid \mathrm{Q}^R\}$, where Q^R is the position after White has played there and Q^L is the position after black has played there. After appropriate changes of markings (which affects only the integer parts of the relative score), we may continue as follows:



$$\mathrm{Q}^R \quad = \quad \{ \quad 0 \quad , \quad ⧿ \quad \mid \quad * \quad , \quad ⧿ \quad \}$$

Although 0 and ⧿ are both formal Black followers of Q^R, the miny option is dominated and we have

$$\mathrm{Q}^R = \{0 \mid *, ⧿\}.$$

The two formal White followers are $*$ and ⧿, which are incomparable. However, White's move from Q^R to ⧿ reverses through a hot position to 0, and we

region	chilled value	Δ^L	Δ^R
A	$0^3 \mid ⧾$	$⧿ \mid 0^3$	$\{0^3 \mid ⧾\}\{⧿ \mid 0^2\}$
B	$1\frac{3}{4} \mid -2$	$3\frac{3}{4} \mid 0$	
C	$-\frac{1}{4}$	$-\frac{1}{4}$	
DH	$\frac{1}{8}$	$\frac{1}{8}$	
E	$ko[1,0]$		
F	$*$	$*$	
G	$\{\frac{5}{8} \parallel ⧿_{1/2} \mid 0\} \mid -2$	$2\frac{5}{8} \mid 2\{⧿_{1/2} \mid 0\} \parallel 0$	
G^L	$\frac{5}{8} \parallel ⧿ \mid 0$	$\frac{5}{8}\{0 \mid ⧾\} \mid 0$	
I	$1 \mid -\frac{1}{2}$	$1\frac{1}{2} \mid 0$	
J	$4 \mid -5$	$9 \mid 0$	
K	$0 \mid -\frac{1}{2}$	$\frac{1}{2} \mid 0$	
M	$\frac{5}{4} \mid -\frac{1}{2}$	$1\frac{3}{4} \mid 0$	
$\mathrm{L}[S^L]$	θ		
N	$0 \mid ⧾ \parallel -\frac{7}{8} \mid -1$	$\{1 \mid \frac{7}{8}\}\{0 \mid ⧾\} \parallel 0$	
N^R	$-\frac{7}{8} \mid -1$	$\frac{1}{8} \mid 0$	
O	$\approx ⧾_{13}$	$\approx ⧿_{13}$	$\approx \{13 \mid 0\}⧾_{13}$
P	$\{12{-}\theta\} \mid \{-12{+}\theta\}$	$\approx 24 \mid 0$	
Q	$2ish \mid -2{\downarrow}*$	$4ish \mid 0$	
$\mathrm{R}[S^L]$	θ		
S	$\{\frac{1}{2}{+}\theta\theta\} \mid -1{\downarrow}{\downarrow}{\downarrow}*$	$1\frac{1}{2}\theta\theta{\uparrow}{\uparrow}{\uparrow}* \mid 0$	
T	$\approx ⧾_6$	$\approx ⧿_6$	$\approx \{6 \mid 0\}⧾_6$
U	$*$	$*$	
V	$2 \parallel \ \mid 0 \mid ⧾_1 \parallel -1*$	$3* \mid 2\{⧿_1 \mid 0\} \parallel 0$	
V^R	$0 \mid ⧾ \parallel -1*$	$1*\{0 \mid ⧾\} \mid 0$	

Table 1. Chilled value and incentives of each region. For most regions, Δ^L and Δ^R are the same. The symbol θ is defined on page 209.

have the canonical form

$$\mathrm{Q}^R = \{0 \mid *, 0\} = {\downarrow}*.$$

To compute Q^L, we begin with the standard assumption that "sufficiently distant stones are safely alive", which in this case suggests that we assume White has already played from I twice to reach I^{RR}. In that environment, it is easily seen that, after an appropriate adjustment of markings, $\mathrm{Q}^{LLL} = ⧿$ and $\mathrm{Q}^L = ⧿ \mid 0^2$.

We then obtain the value of

$$\mathrm{Q} = \{2\{\leftarrow \mid 0^2\} \mid -2{\downarrow}{*}\}.$$

Other values shown in Table 1 are computed by similar calculations.

In general, values of games may be conveniently partitioned into three sets: cold, tepid, and hot. Cold values are numbers. In Table 1, the regions with cold values are DH and C. Tepid values are infinitesimally close to numbers. In Table 1, tepid values are A, F, L, O, T, and U. The other values shown in Table 1 are hot. In general, hot moves, tepid moves, and cold moves should be played in that order. So, to determine the first phase of play, we will need to determine the order to play the hot moves in Table 1.

5. Incentives

As explained in Section 5.5 of *Mathematical Go*, a useful way to compare different choices of moves is to compute their incentives. In general, the game

$$Z = \{Z^L \mid Z^R\}$$

has a set of Left incentives of the form

$$\Delta^L(Z) = Z^L - Z,$$

where there are as many Left incentives as there are Left followers Z^L. Similarly, the Right incentives of this game have the form

$$\Delta^R(Z) = Z - Z^R.$$

Calculations of canonical incentives often require considerable attention to mathematical detail. For example, consider a generic three-stop game

$$G = \{x \parallel y \mid z\}$$

where x, y, and z are numbers with $x > y > z$. Then

$$\begin{aligned} \Delta^L(G) &= G^L - G \\ &= x - \{x \parallel y \mid z\} \\ &= x + \{-z \mid -y \parallel -x\} \\ &= \{x-z \mid x-y \parallel 0\} \end{aligned}$$

by the *Number Translation Theorem* on page 48 of *Mathematical Go*. But

$$\begin{aligned} \Delta^R(G) &= G - G^R \\ &= \{x \parallel y \mid z\} + \{-z \mid -y\} \\ &= \{\{x-z \mid x-y\}, \{x-z \parallel y-z \mid 0\} \mid 0, \{x-y \parallel 0 \mid z-y\}\}. \end{aligned}$$

Deleting Right's dominated option gives

$$\Delta^R(G) = \{\{x-z \mid x-y\}, \{x-z \parallel y-z \mid 0\} \mid 0\}.$$

The second Left option is dominated if $x-y \geq \{y-z \mid 0\}$ or, equivalently, if $x-y \geq y-z$, or if $x+z \geq 2y$. In that case, $\Delta^R(G) = \Delta^L(G)$. However, if $x-y < y-z$, then $\Delta^R(G) > \Delta^L(G)$. Furthermore, if $2(x-y) < y-z$, then

$$\text{temperature}(\Delta^R(G)) \geq \text{temperature}(\Delta^L(G)).$$

Most of the positions shown in Table 1 have a single canonical Left incentive and a single canonical Right incentive. Although these two are often formally different, further calculation often reveals their values to be equal. In Table 1, this happens on all rows except A, E, L, O, R, and T.

If Black plays from S to S^L, he converts each of the regions L and R into a Black kothreat. Although each of these kothreats has a mathematical value of zero, it can, of course, be used to affect the outcome of the ko. We denote each such kothreat by the Greek letter θ, because this letter looks somewhat like 0 and because "theta" and "threat" start with the same sound. Similarly, a White kothreat can be denoted by $-\theta$.

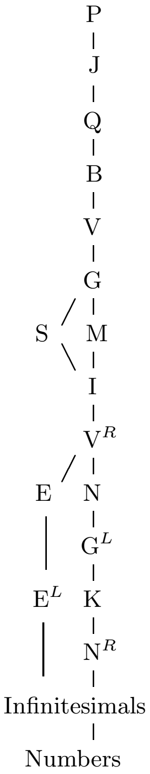
Incentives are games. Just like any pair of games, a pair of incentives may be comparable or incomparable. Fortunately, it happens that most pairs of incentives shown in Table 1 are comparable. Except for a possible infinitesimal error in one case (another "nit" that will be picked away much later), the incentives can be conveniently ordered as shown in Table 2. The top portion of this table shows the positions P, J, Q, ... , N^R, each of which has a unique incentive, which is the same for both Left and Right, and which is greater-ish than the incentives of all positions listed below it and less-ish than incentives of all positions listed above it.

Notice that some positions with nonzero values, such as minies or tinies, may also serve as kothreats for Black or White, respectively. Although their *values* are infinitesimal or zero, tinies, minies, and θ's of either sign, all of them have hot *incentives* for at least one player, and so we will need to include them in our studies of "hot" moves.

Either player will always prefer to play a move with maximum incentive. So, (ignoring a potential infinitesimal nit) each player will prefer to play on P, J, Q, B, ... , in the order listed in the top portion of Table 2.

There are some hot moves that cannot be so easily fit into the ordering in the top portion of Table 2: the ko at E, potential White kothreats at T and O, potential Black kothreats at P^R, $\mathrm{R}[\mathrm{S}^L]$, and $\mathrm{L}[\mathrm{S}^L]$, and the position at S, whose play creates or destroys two Black kothreats. These moves are listed in the bottom portion of Table 2, in the order in which they are expected to be played: S will be played before the ko; once the kofight starts, kothreats will alternate; and finally someone will eventually fill the ko.

Good play must somehow intersperse the ordered set of moves shown at the bottom of Table 2 into the ordered set shown at the top. In order to do this, it is very helpful to consider the *pseudo-incentives* of E and S.



region	Δ	I	II	III	IV	V
P	$\approx 24 \mid 0$	1	1	1	1	1
J	$\approx 9 \mid 0$	2	2	2	2	2
Q	$4ish \mid 0$	3	3	3	3	3
B	$3\frac{3}{4} \mid 0$	4	4	4	4	4
V	$3* \mid 2\{\vdash_1 \mid 0\} \parallel 0$	5	5	5	5	5
G	$2\frac{5}{8} \mid 2\{\vdash_{1/2} \mid 0\} \parallel 0$	6	6	6	6	6
M	$1\frac{3}{4} \mid 0$	8	8	8	7(?)	7(?)
I	$1\frac{1}{2} \mid 0$	9	9	9	9	9
V^R	$1*\{0 \mid \underline{+}\} \mid 0$	10	10	10	10	10
N	$\{1 \mid \frac{7}{8}\}\{0 \mid \underline{+}\} \parallel 0$	11	11	11	11	12
G^L	$\frac{5}{8}\{0 \mid \underline{+}\} \mid 0$	13	22	12(?)	13	13
K	$\frac{1}{2} \mid 0$	15	24	13	14	14
N^R	$\frac{1}{8} \mid 0$	16	25	24	31	
S	$1\frac{1}{2}\theta\theta\Uparrow\Uparrow* \mid 0$	7	7	7	8	8
E	ko	12	12	14	12	
E^L	$-ko$				15	
$R[S^L]$	$\Delta\theta$				16	
E	ko				18	
T	$-\Delta\theta$		13(?)	15	19	
E^L	$-ko$		15	17	21	
P^R	$\Delta\theta$		16	18	22	
E	ko		18	20	24	
O	$-\Delta\theta$		19	21	25	
E^L	$-ko$		21	23	27	
$L[S^L]$	$\Delta\theta$				28	
E	ko				30	
E^L	$-ko$					
kofill		14	23	25	32	11
totals		0	$0 \mid \underline{+}$	$-\frac{3}{8}ish$	$\frac{1}{8}ish$	$\frac{1}{8}ish$

Table 2. Sorted incentives (Δ) for the regions, and the dominant lines of hot play. Columns I–V indicate different possibilities for the order in which the moves can be made (see Section 7). Regions of Table 1 that have no hot incentives are omitted.

6. Pseudo-Incentives Related to the Ko at E

Suppose we have a set of games A, B, C, $\ldots$, K, L, $\ldots$ with ordered incentives $\Delta A > \Delta B > \Delta C > \cdots > \Delta K > \Delta L > \cdots$. Suppose also that there is a kofight in progress and that White has six more kothreats, $-\theta[6]$, $-\theta[5]$, $\ldots$, $-\theta[1]$. Then a competently played game will continue as follows:

Black: kotake	1	7	13	19	25	31
White: threat	2: $-\theta[6]$	8: $-\theta[5]$	14: $-\theta[4]$	20: $-\theta[3]$	26: $-\theta[2]$	32: $-\theta[1]$
Black: response	3	9	15	21	27	33
White: kotake	4	10	16	22	28	34
Black:	5: A	11: C	17: E	23: G	29: I	35: K
White:	6: B	12: D	18: F	24: H	30: J	36: kofill

Notice that White declines to fill the ko at moves 6, 12, 18, 24 or 30. He eventually fills the ko at move 36, because he has no more kothreats. In the above situation, if X is any canonical value such that $X = \Delta X$ and that $\Delta K > \Delta X > \Delta L$, we may treat the ko as if its value were X. Such treatment gives the correct time at which Right will fill the ko. And both ko and X contribute the same amount to the outcome of the game, since $ko^R = X^R = 0$. For these reasons, we write $ko \equiv X$ and we say that the ko has a *pseudo-incentive* $\Delta^R(ko) = \Delta(X)$.

This viewpoint is especially helpful when we can predict an appropriate value of X from global considerations. Suppose, for example that E, the ko in this problem, appears in an environment containing a few pairs of games with incentives between $\{1 \mid 0\}$ and $\{0 \mid 0\}$, more White kothreats, and even more infinitesimals. Then we claim that $ko^R = 0ish$, $ko^L = \{1 \mid 0\}ish$, $\Delta^R(ko) = 0ish$, $\Delta^L(ko) = \{1 \mid 0\}ish$.

Left begins fighting the ko as soon as she has no more moves of incentive $\{1 \mid 0\}ish$ or greater. Right eventually fills the ko when he runs out of kothreats, which happens during the play of infinitesimals.

In the present problem, there are at least 2 and at most 4 moves with incentives between $\{1 \mid 0\}ish$ and $0ish$: N, G^L, K and N^R. There are many potentially tepid moves, on games such as A, F, G^{LR}, N^L, Q^R, S^R and U, V^{RL}. So, if there are 2 net White kothreats, then $\mathrm{E} \equiv 0ish$. Similarly, if there are 2 net Black kothreats, then $\mathrm{E}^L \equiv 1ish$ and $\mathrm{E} \equiv \{1 \mid 0\}ish$. In either case, we obtain a useful bound on the pseudo-incentives of E, namely,

$$\Delta\mathrm{E} \leq \{1 \mid 0\}ish.$$

This allows us to assume that canonical players prefer to play on $\mathrm{V}^R ish$ rather than on E.

A continuation of this argument allows us to obtain an upper bound on the incentive of $(\mathrm{S} + \mathrm{E})$.

Let $Z = \{\mathrm{S}^L + \theta\theta \mid \mathrm{S}^R - \theta\theta\}$. Then, if Z is played by Left, there are two net Black kothreats and $\mathrm{E} \equiv \{1 \mid 0\}ish$, but if Z is played by Right, there are two net White kothreats and $\mathrm{E} \equiv 0ish$. So

$$\Delta(Z + \mathrm{E}) = \Delta(\{1\tfrac{1}{2} \mid \tfrac{1}{2} \parallel -1\}ish) = \{2\tfrac{1}{2} \mid 1\tfrac{1}{2} \parallel 0\}.$$

Since Z is always played before E, we have $\Delta(Z+\mathrm{E}) = \Delta(Z)$; and since kothreats are at worst harmless, we have the following bounds on pseudo-incentives:

$$\Delta\mathrm{S} < \Delta Z < \{2\tfrac{1}{2} \mid 1\tfrac{1}{2} \parallel 0\}.$$

We may now obtain bounds on the incentive of $\mathrm{S} = \{\frac{1}{2} \mid -1\}ish$. Since Black kothreats can not harm Black, we have

$$\Delta\mathrm{S} \geq \Delta(\tfrac{1}{2} \mid -1) = \{1\tfrac{1}{2} \mid 0\}.$$

Thus,

$$\{2\tfrac{1}{2} \mid 1\tfrac{1}{2} \parallel 0\} \geq \Delta\mathrm{S} \geq \{1\tfrac{1}{2} \mid 0\}.$$

7. The Search

Using the bounds on the pseudo-incentives of S and E, we obtain the partial ordering shown at the left of Table 2. Using this partial ordering, canonical players face only two choices: whether to play S before or after M, and how to play the ko at E.

In each column of Table 2, the entries show the move numbers on which that row is played.

The choice between S and M occurs at move 7. The dominant continuations following 7S appear in columns I, II and III; the dominant continuations following 7M appear in columns IV and V. But both players may need to decide when to play their first move at E or E^L. After 7M, White later chooses between 11E (column V) and 11N (column IV). After 7S, White clearly does better not to fill a ko that he can win later, so he plays 11N and Black may then choose between $12\mathrm{G}^L$ (column III) and 12E (column I and II). After 12E, White can still choose either to fight the ko with 13T (column II) or not (column I).

We assert that the canonical solution is column I, which corresponds to the game played in Figure 3.

The bottom row of Table 2 shows the total value of the outcome of each line of play, *relative to the line of play shown in column* I. Each of these values can be easily computed, using only the information shown in Table 1. For example, to compare the outcome of column III with the outcome of column I, we tally the rows where the entries have different parities. In row G^L, White (odd numbers) played in column I, but Black (even numbers) played in column III, and so that row's contribution to the difference is

$$(\mathrm{G}^L)^L - (\mathrm{G}^L)^R = 2\Delta(\mathrm{G}^L) = \tfrac{5}{8}ish.$$

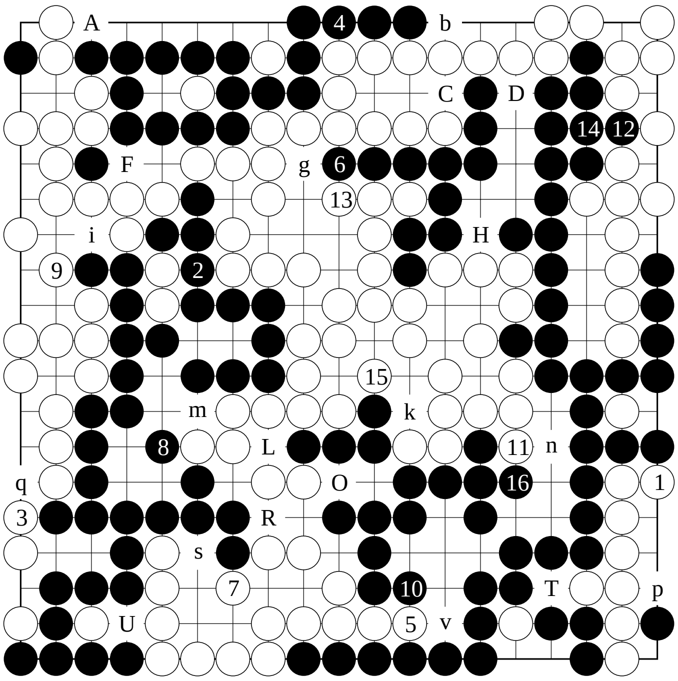


Figure 3. Canonical play of hot regions: moves 1–16.

The only other row that contributes any difference is the *kofill*. The raw (warmed) difference in scores depending on who eventually fills the ko at E is 4 points (one for the stone captured in the ko itself, and three points of white territory along the top right edge of the board). But there is also a difference of three moves, and so the chilled value of this ko is 1 point. Thus, the "kofill" line contributes -1 point to the total. Then the bottom line of column III is obtained as $\frac{5}{8}ish - 1 = -\frac{3}{8}ish$.

The calculation of the bottom line for column V is only slightly more subtle:

$$\begin{aligned}
-2\Delta(\mathrm{M}) &= -1\tfrac{3}{4}\\
+2\Delta(\mathrm{N}) &= \{1 \mid \tfrac{7}{8}\}ish\\
+2\Delta(\mathrm{K}) &= +\tfrac{1}{2}\\
-1\Delta(\mathrm{N}^R) &= \{0 \mid -\tfrac{1}{8}\}\\
+2\Delta(\mathrm{S}) &= +1\tfrac{1}{2}ish\\
\text{kofill} &= -1\\
\hline
\text{relative total} &= +\tfrac{1}{8}ish
\end{aligned}$$

The totals of other columns are computed by the same method. From the bottom row, we deduce that the moves followed by question marks in columns III, IV and V are suboptimal.

Although the difference between column II and column I is infinitesimal, $\{0 \mid +\}$ is positive. Furthermore, Black gets the next (tepid) play in column II, whereas White gets the next (tepid) play in column I. So column II is decidedly better for Black then column I.

This completes our discussion of Table 2 and its conclusion, which is that for the first 16 moves, an optimal canonical line of play is as shown in Figure 3.

8. Verification of Robustness

Before continuing with any further play, we now pause to verify that the dominant column of Table 2 is unaffected by the removal of all of the simplifying assumptions we have made so far. Each of our simplifying assumptions is now seen as a "nit" to be picked out. In any problem of this type, we hope that our intuitive assumptions are benign in the sense that our conclusions survive their extraction. If this proves false, the "nit" is no longer a nit, but a cause to redo the prior partitioning and calculations in a more accurate way, even if that entails a significant increase in complexity.

Many Go players may find our nitpicking attention to mathematical detail distasteful. We view it as a small price to pay for the rigor that results.

There are two basic types of nits:

- Some sets of adjacent regions are dependent.
- Some incentives in the top part of table 1 are not quite strictly ordered.

In the present pair of problems, we have thus far included five nits:

8.1. Nit 1: Q and I are dependent. Let Q and I be the independent idealized regions obtained by immortalizing the string of six white stones just below the letter I, and let W be the actual region that is the union of these regions, including the string of six white mortal white stones between them. Then, by playing the difference games, it can be verified that

$$\mathrm{W} \geq \mathrm{I} + \{2\{\dashv \mid 0^2\} \mid -2{\downarrow}{*}\} \qquad \text{and} \qquad \mathrm{W} \leq \mathrm{I} + \{2\dashv \mid -2{\downarrow}{*}\}.$$

Hence, $\mathrm{W} = \mathrm{I} + \{2ish \mid -2{\downarrow}{*}\}$. Thus no errors result from setting $\mathrm{W} = \mathrm{I} + \mathrm{Q}$ if Q is taken as $2ish \mid -2{\downarrow}{*}$.

8.2. Nit 2: N and V are dependent. There is a slight interaction between the regions N and V shown in Figure 1. For example, in the presence of O^{RL}, the chilled value of $\mathrm{N}^{RL} + \mathrm{V}^{RRR}$ is $\uparrow$, instead of the value implicit in Table 1, which is $\frac{1}{8}$. This nit is closely related to Nit 3.

8.3. Nit 3: $\Delta(\mathrm{N})$ is not strictly less than $\Delta(\mathrm{V}^R)$. Let N and V be the independent idealized regions obtained by immortalizing the black string just left of N and just above V; let X be the actual region that is the union of these regions, including the mortal black stones between them. We have seen that

$$\begin{aligned}
&\mathrm{N} = 0 \mid \text{⧾}_1 \parallel -\tfrac{7}{8} \mid -1,\\
&\mathrm{V} = 2 \parallel \ \mid 0 \mid \text{⧾}_1 \parallel -1{*},\\
&\mathrm{X} \le \mathrm{V} + \mathrm{N}.
\end{aligned}$$

But, by playing the difference game, it can be verified that $\mathrm{X} \ge \mathrm{V} + \mathrm{Y}$, where $\mathrm{Y} = 0 \mid \text{⧾}_1 \parallel -1\uparrow \mid -1$. We notice that

$$\Delta(\mathrm{V}^R) > \Delta(\mathrm{Y}) > \Delta(\mathrm{N}),$$

from which we conclude that all plays on X will be taken in the same order as the corresponding plays on (V + N). This eliminates the concern of Nit 3.

To address Nit 2, we observe that no difference can arise between X and $(\mathrm{V} + \mathrm{N})$ unless N is played to N^{RL}, which happens only in columns I and III of Table 2. In both of these cases, V is played to V^{RL}. Under these circumstances, the (V + N) has been played to

$$(\mathrm{V} + \mathrm{N})^{RLRL} = \mathrm{V}^{RL} + \mathrm{N}^{RL} = \{0 \mid \text{⧾}_1\} - \tfrac{7}{8},$$

whereas X has been played to

$$\mathrm{X}^{RLRL} = \{0 \mid \text{⧾}_{1\frac{1}{8}\downarrow}\} - \tfrac{7}{8}.$$

Thus, the concern of Nit 2 is eliminated by suppressing the subscript of the tiny that appears in V^{RL}.

8.4. Nit 4: J and Mare dependent. Let Z be the union of the dependent regions J and M. By careful calculation, it can be shown that, to within *ish*,

$$\Delta^L\mathrm{Z} = \Delta^R\mathrm{Z} = \Delta\mathrm{Z} = \{9 + \{\tfrac{5}{4} \mid -\tfrac{1}{2}\} + \{1 \mid -1\} \parallel 0\}$$

whence

$$\{10 \mid 0\} > \Delta\mathrm{Z} > \{9 \mid 0\}.$$

Hence, ΔZ fits into the total rank ordering of Table 2 in precisely the same position as ΔJ. It follows that, in the dominant line of play, Black move 2 will change Z to $\mathrm{Z}^L = \mathrm{J}^L + \mathrm{M}$, because J^L and M are independent.

8.5. Nit 5: T and P are dependent. Mathematically, this dependence is relatively trivial. It affects only the subscript of the tiny value at T, and thus has no affect on Table 2.

However, this dependence underlies a very interesting opening move: White 1 at T. Although Black can then win, her chilled margin of victory is only infinitesimal, and so great care is required. Black can play 2 at T^R. Then play continues as in column II (delayed by 2 moves), with White 3 at P, Black 4 at J, ... , until White plays move 13 at N. Black then plays 14G^L. If White neglects to fill the ko on move 15, then Black plays E at move 16. She then fights the ko until she wins and fills it, attaining a bottom-line score on the hot regions of $+\frac{1}{2}$*ish* or better. If instead White plays 15E, then play continues as in column II ending with a positive value on the bottom line.

Having now extracted all known nits, we now resume our investigation of how play should continue from Figure 3.

9. Tepid Regions

A general theorem in (ko-free) Combinatorial Game Theory states that one need never play a tepid move if one can play a hot move instead. This theorem suggests a natural breakpoint in endgame play, roughly corresponding to the break between the moves shown in Figure 3 and the moves shown in subsequent figures. For convenience of exposition, we assume that White continues play by eliminating the tinies at O and T at moves 17–20. In fact, these moves might occur either earlier or later.

Omitting the integer-valued markings, and excepting the tinies, which have no atomic weight, the tepid regions of Figure 3 are tabulated in Table 3.

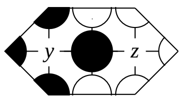
region	value	atomic weight
A	$0^4 \mid$ ⧾	$+4$
v	$0 \mid$ ⧾	$+1$
g	$\vdash \mid 0$	-1
q	$\downarrow *$	-1
s	$\downarrow\downarrow\downarrow *$	-4
F	$*$	0
u	$*$	0

Table 3. Atomic weights.

In general, if no hot moves are available, Black will strive to increment the atomic weight; White, to decrement it. To this end, Black will typically attack regions of negative atomic weight, while White will attack regions of positive atomic weight.

The total of the atomic weights in Table 3 is -1, and so White, moving first, has a relatively straightforward win.

In the Japanese rules, Black will soon resign. However, under the Ing rules there is a flaw in this analysis, which is shown in the picture of s^{LLLLL} on the right.



Using the Ing rules or any other variation of Chinese rules, an assertive Black may hope to score a point at y. Even after White has played at z, Black may claim one point of Chinese territory at y. To uphold such a claim, Black will need sufficiently many kothreats that she can avoid filling y until after all dame are filled and White is forced to either pass or fill in a point of his own territory while the node y remains empty and surrounded by Black. So we have two different views of chilled values:

Chinese scoring with many θ’s	Japanese scoring
$\mathrm{s}^{LLLLL} = 1$	$\mathrm{s}^{LLLLL} = 0$
$\mathrm{s} = 1 \mid 0^5$	$\mathrm{s} = 0 \mid 0^5$
$= \mathord{\vdash\!\dashv} \mid 0^3$	$= \downarrow\downarrow\downarrow\downarrow\ast$

So Black’s best hope is to play the tepid regions under the assumption that she can win a terminal Chinese kofight. White must play to win under Japanese rules, because if White fails to win under Japanese rules, the Chinese kofight will not help him. According to a theorem of David Moews, the optimal sequence of play is given by the ordering explained in Appendix E of *Mathematical Go*. This canonical line of play is shown in Figure 4.

The result is that

$$\{0^4 \mid \text{⧾}\} + \downarrow\ast + \downarrow\downarrow\downarrow\downarrow\ast < 0$$

but

$$\{0^4 \mid \text{⧾}\} + \downarrow\ast + \{\mathord{\vdash\!\dashv} \mid 0^3\} > 0.$$

Since the subscript on the tiny in game A exceeds the subscripts in games v and g, these inequalities remain valid when applied to the sum of all tepid regions of Figure 3. No matter who plays next, Black can win the game if $\mathrm{s}^{LLLLL} = 1$, but White can win if $\mathrm{s}^{LLLLL} = 0$.

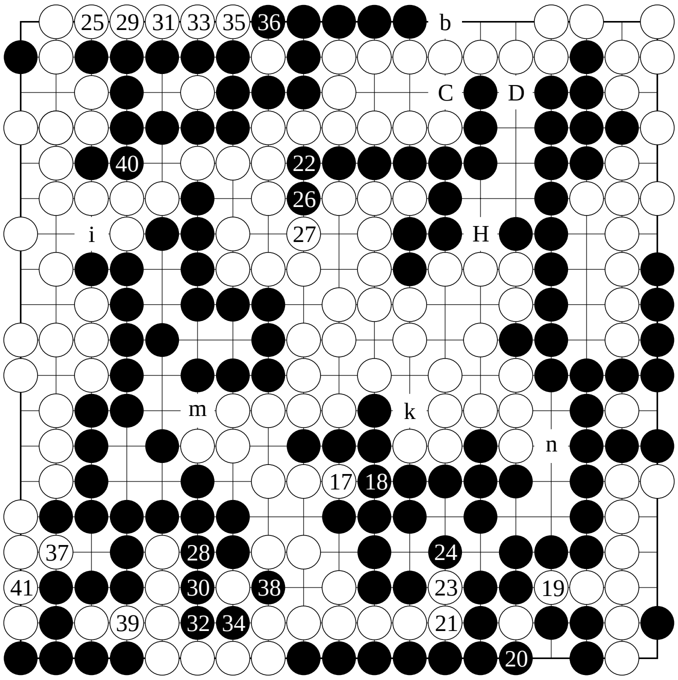


Figure 4. Play of tepid regions: moves 17–41.

The game should therefore continue as shown in Figure 4. Moves 21–36 all have incentive of at least $\uparrow *$. After move 36, White's view of the total value is $\downarrow$, from which White can win only by playing at 37, because it is the only move on the board with incentive $\downarrow$. Black's 38 is hot in her view, but in White's view the incentive of Black's move 38 is only $*$. Moves 39, 40, and 41 are all stars.

White's 41 is the last tepid move. This milestone of play occurs at the conclusion of Figure 4 and the start of Figure 5. The number avoidance theorem mentioned on page 48 of *Mathematical Go* ensures that this same milestone occurs in all well-played combinatorial games.

10. Play of Numbers

The seven regions that remain active at the conclusion of Figure 4 all have values that are numbers. Subsequent play on each of these numbers can effect not only its mathematical value, but also the kothreats available for the Chinese kofight that will occur later. In this kofight, dame will prove helpful to White. The number of such dame can depend on how the numbers are played. For example, consider a marked and chilled position such as

$$\mathrm{n}^{RR} + \mathrm{C}^{L} = \tfrac{1}{2} - \tfrac{1}{2} = 0$$

Either player, moving second on this pair of regions, can ensure that the total score is zero. However, the player who moves first can control the number of pairs of dame. If White goes first, he plays on C^L, thereby obtaining one pair of dame. If Black goes first, she also plays on C^L, thereby preventing any dame. Evidently, C^L is a number on which both players are *eager* to play; n^{RR} is a number on which they are *reluctant*. Typically, since White desires dame and Black dislikes them, numbers with positive values will be eager and numbers with negative values will be reluctant. However, local configurations can make exceptions to this rule, as happens in position i. Even though its value is $-\frac{1}{2}$, both players are *eager* to move there. That is because if Black manages to play to i^{LL}, she acquires two kothreats along the left side. So Black desires to move on i in hopes of progressing toward that goal, and White desires to move on i in order to stop her. The numbers are thus partitioned as follows:

eager numbers	reluctant numbers
$\mathrm{DH} = \frac{1}{8}$	$\mathrm{b} = -\frac{1}{4}$
$\mathrm{n} = \frac{1}{8}$	$\mathrm{C} = -\frac{1}{4}$
$\mathrm{m} = \frac{1}{4}$	$\mathrm{k} = -\frac{1}{2}$
$\mathrm{i} = -\frac{1}{2}$	

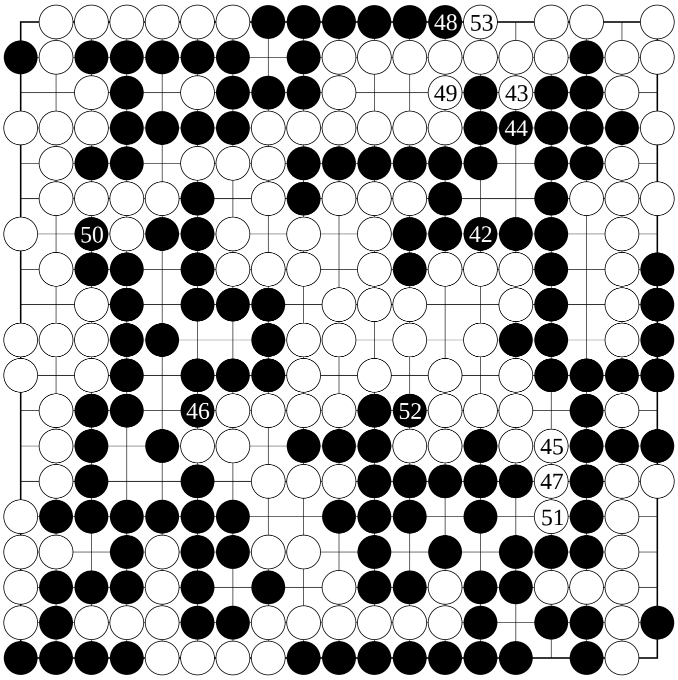


Figure 5. Play of numbers: moves 42–53.

This tabulation provides the rationale for the sequence of play displayed in Figure 5. Notice how the incentives decrease monotonically as play progresses:

formal incentive	moves
eager $-\frac{1}{16}$	42–43
eager $-\frac{1}{8}$	44–45
eager $-\frac{1}{4}$	46–47
reluctant $-\frac{1}{4}$	48–49
eager $-\frac{1}{2}$	50–51
reluctant $-\frac{1}{2}$	52–53

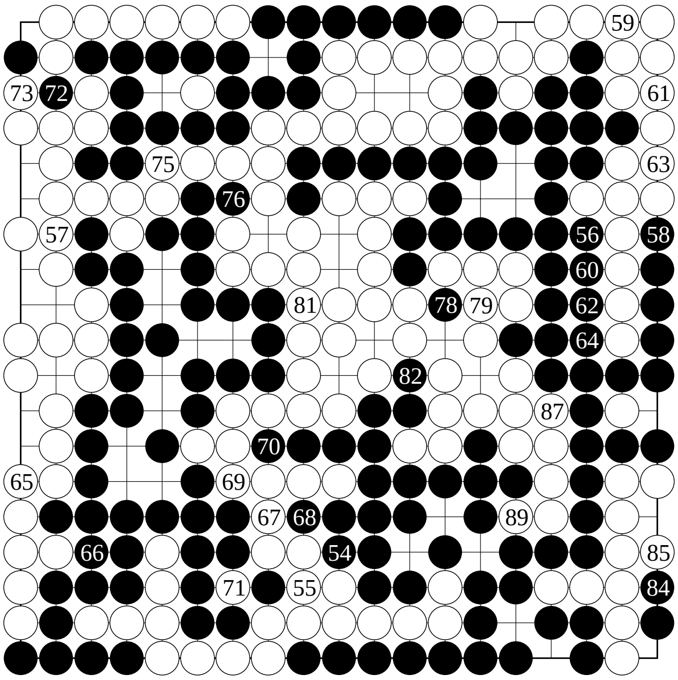
By partitioning the eager and reluctant numbers into separate subsystems, each of which sums to an integer, White can play each integer independently and thereby ensure that he gets at least one pair of dame. However, Black gets everything else. At each turn, she gets to choose among at least two alternatives, and White is then forced to take whichever is left over. Black blocks a potential White kothreat by playing at 46 rather than at 47. Black's move 50 gives her the opportunity to play 58 later in Figure 7.

If White plays 47 at 48 or 49, or if he plays 51 at 52 or 53, there would be two fewer dame at the conclusion of play. In Figure 5, any of these errors would cost White the game.

Move 53 marks the end of the game in Japanese scoring, but in Chinese scoring, lots of action remains.

11. The Chinese Kofight

Figure 6 shows one correct line of play for the completion following Figure 5, in which each player strives to win the game but ignores the secondary issue of how many moves are played until the game ends.



71, 74, 77, 80, 83 and 86: Take the Ko
88: Fill the Ko

Figure 6. Play of Chinese Ko: moves 54–89.

The reader who is familiar only with Japanese scoring should begin by studying the final stage of the fight, which begins after move 70. As White takes the ko at move 71, Black has 3 kothreats, but White has 6 dame. Each pair of dame provides a kothreat for White. After White's move 87, Black has no kothreats. Only one dame remains, but if Black plays it, then White wins the ko that gives him a margin of victory of 3 points. So Black fills the ko at move 88, and White then fills the last dame to win the game by one point.

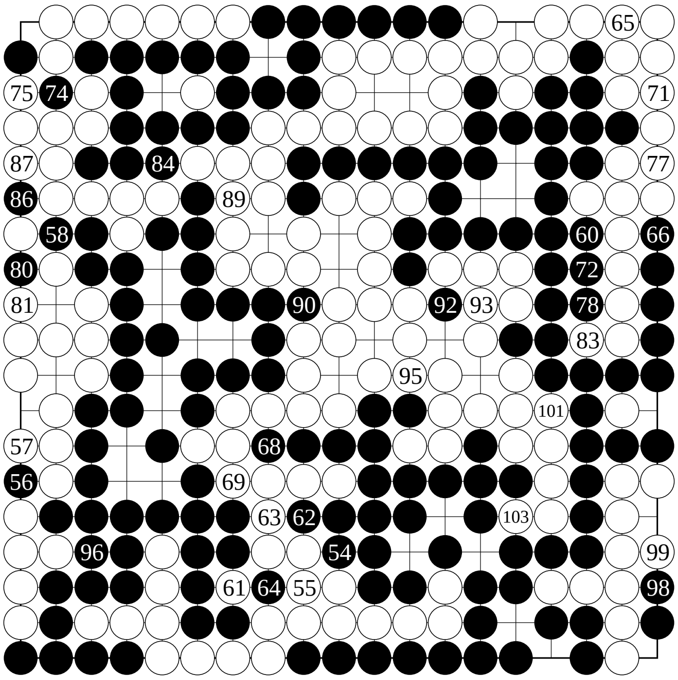
Suppose, however, that there were no dame at locations 75 and 81, and that the last dame was filled by Black's move 83. In that case, after Black's ko capture at move 85, White could do no better than pass, and Black would win the game by one point.

The unusually wide variety of kothreats and potential kothreats shown in Figures 4 or 5 makes this a surprisingly interesting game. The mathematical tools to handle such games are still being developed, and so the reader may enjoy working out the present problem by the traditional Go-playing method of reading out the various lines of play.

It is somewhat surprising that each player can play this portion of the game in several different ways, all of which lead to equivalent results. From the position shown after move 53 in Figure 5, White can win. However, if this position contains two fewer dame, then Black could win. The variation of the position that contains two fewer dame might arise either because the starting position was Problem 2 rather than Problem 1, or because in Problem 1 White failed to play the numbers on moves 42 to 53 with sufficient care.

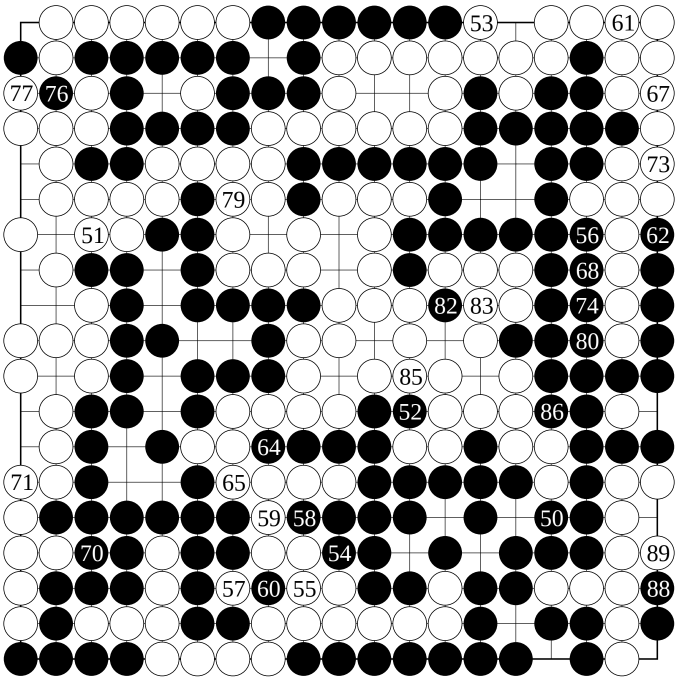
Either player can elect to prolong the Chinese kofight.

Figure 7 shows a variation in which Black, unable to win, forces White to



59 at 56
64, 67, 70, 73, 76, 79, 82, 85, 88, 91, 94, 97 and 100: Take the Ko
102: Fill the Ko

Figure 7. Black stalls his loss: moves 54–103.



57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87 and 90: Take the Ko

Figure 8. Black wins Problem 2 quickly: moves 50–90.

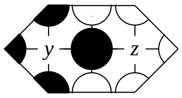
play until move 103. This line of play includes Black's surprising move 56, which sacrifices a kothreat prematurely in order to lengthen the game. A slightly less brilliant line of play by Black would eliminate moves 56 and 57, leading to a White win after 101 moves instead of 103.

Figure 8 shows a conclusion to Problem 2, in which Black can prevail, but apparently no sooner than move 90. To achieve this rapid victory, Black deviates from prior lines of play at move 50. There are other ways in which Black can attempt to expedite his victory by deviating from canonical play, but we think that none of them succeed. For example, if Black postpones playing at 22 until after he has played at 38, then White can play at 41 instead of at 37, after which Black cannot win because his early kothreat never materializes.

12. The $1,000 Dame

It is tempting to think that all of our prior analysis of Problems 1 and 2 should also apply to Problems 3 and 4. Canonical play for the first 53 moves is indeed the same for all four problems. Reading further ahead, it is very plausible to foresee the same Chinese kofight. Eventually, the number of Black's threats will exceed the number of pairs of dame and so for several weeks we believed that Problem 4 behaved like Problem 2: Black should win in about ninety moves.

However, Kuo-Yuan Kao then pointed out a fatal flaw in this reasoning, which originates at s^{LLLLL}:



If White plays at z, while other "dame" remain on board, then Black has enough kothreats to fight and win the Chinese ko at y. Eventually, White runs out of other dame to fill and he is forced to pass or fill in his own territory. Only after White has done that can Black afford to fill y and still win the game.

Mathematically, after Black plays y and z remains vacant the unchilled value is $*$. But, after White plays z and y remains vacant, then if Black can win the Chinese Ko, its unchilled Chinese value is 1. So the original position just shown has unchilled value equal to $\{* \mid 1\} = 0$.

This equation reveals how White can win Problem 4. After canonical play through move 53, he should refuse to play at z. Black and White both play dame elsewhere until no other dame remain. White then has no kothreats and Black has many, but they do her no good, because both y and z are still vacant and there are no other dame anywhere else on the board. On move 73, White fills the last dame elsewhere. If Black then fills y, White fills z and wins by one point. If Black passes at move 74, White fills z and still wins by one point in Ing scoring. Black gets her point at y (either filled or unfilled), but she still loses the game.

So, thanks to the $1,000 "false dame" at z, we believe that White wins Problem 4. If he plays canonically for the first 53 moves, he eventually fills the last dame on move 75. But Black then prolongs the game another ten moves by playing out each of her five kothreats.

Problem 3 is an easier version of Problems 1 and 4.

The same argument reveals why, in Problem 1, in Figure 6 Black cannot afford to play Black 54 anywhere else. If instead she played Black 54 at 57, White could play 55 at 54 and then refrain from playing at 55, thereby winning the game by avoiding the Chinese kofight.

13. The Non-Canonical Surprise

Virtually all of the contestants in Ishi Press's Thousand-Dollar Ko competition made fatal mistakes on early moves. But among these weak lines of play, two contestants uncovered a real gem:

1P, 2J, 3Q, 4B, 5S!

From the partial order chart at the left of Table 2, this looks hopelessly wrong. After all, $\Delta S < \Delta G < \Delta V$.

So we played the continuation: 1P, 2J, 3Q, 4B, 5S, 6V, 7G, and reached the same position that might also have arisen from 1P, 2J, 3Q, 4B, 5G, 6V, 7S. This

region	Δ	VI	VII	VIII
B	$3\frac{3}{4} \mid 0$	4	4	4
V	$3* \mid 2\{⧿_1 \mid 0\} \parallel 0$	6	6	6
G	$2\frac{5}{8} \mid 2\{⧿_{1/2} \mid 0\} \parallel 0$	5	5	5
M	$1\frac{3}{4} \mid 0$	8	8	8
I	$1\frac{1}{2} \mid 0$	9	9	9
V^R	$1*\{0 \mid ⧾\} \mid 0$			
N	$\{1 \mid \frac{7}{8}\}\{0 \mid ⧾\} \parallel 0$	11	20	10(?)
G^L	$\frac{5}{8}\{0 \mid ⧾\} \mid 0$			
K	$\frac{1}{2} \mid 0$	13	22	11
N^R	$\frac{1}{8} \mid 0$	14		
S	$1\frac{1}{2}\theta\theta\uparrow\uparrow\uparrow\uparrow* \mid 0$	7	7	7
E	ko	10	10	12
T	$-\Delta\theta$		11(?)	13
kofill		12	21	23
totals		$0ish$	$\frac{3}{8}ish$	$-\frac{1}{8}ish$

Table 4. An extension of Table 2.

less dramatic sequence focuses our attention on an intriguing inequality,

$$\Delta\mathrm{G} < \Delta\mathrm{V},$$

whose truth depends on both of the following true assertions:

$$2\tfrac{5}{8} < 3\,* \qquad \text{and} \qquad \{\tfrac{1}{2} \mid 0 \parallel 0\} = ⧿_{1/2} < ⧿_1 = \{1 \mid 0 \parallel 0\}.$$

So this line is definitely *not* canonical. White's 5G is dominated by the canonical 5V. However, our investigations of continuations of this sequence soon led to the extension of Table 2 shown in Table 4.

The canonical continuation after 5G 6V is evidently as shown in column VI. After the hot games are played, the value is 0-*ish*. The difference between the precise values at the bottom of column VI and column I is

$$\{⧿_1 \mid 0\} + \{0 \mid ⧾_{1/2}\} > 0.$$

It is easy to see that this infinitesimal difference, of atomic weight zero, evaporates near the beginning of the tepid phase of play. The continuations of columns VI and I quickly converge to the same plays of regions A, s, and q, the same careful play of numbers, and the same Chinese kofight. Additional study verifies that there is no difference in Chinese kothreats or dame. The only difference is that White gets there 8 moves sooner via the line of play shown in column VI. So, if White seeks to win as quickly as possible, column VI is his best opening for Problem 1. To stall his defeat in Problem 2, he should open instead as in column I. Black has a superb response to those White contestants who played the surprise opening to Problem 1:

1P, 2J, 3Q, 4B, 5S!, 6G!!

To address the question of whether this problem has any other surprise openings, one can re-examine the partial ordering of incentives shown at the left of Table 2. The inequalities shown in this diagram may be partitioned into strong inequalities and weak inequalities, where an inequality of the form $\Delta \mathrm{W} < \Delta \mathrm{X}$ is *strong* if there exist numbers y and z such that;

$$2\Delta \mathrm{W} < y < z < 2\Delta \mathrm{X}.$$

An order of play that transposes two consecutive moves related by a strong inequality is not only non-canonical; it loses the game. So the search for more winning non-canonical lines of play must violate only weak inequalities in the partial order to the left of Table 2. The only inequalities that might be weak are these:

$$\begin{array}{ccc} \Delta \mathrm{V} > \Delta \mathrm{G} & \Delta \mathrm{V}^R > \Delta \mathrm{E} & \Delta \mathrm{E} > \mathit{ishes} \\ \Delta \mathrm{S} > \Delta \mathrm{I} & \Delta \mathrm{V}^R > \Delta \mathrm{N} & \Delta \mathrm{E}^L > \mathit{ishes} \end{array}$$

The player who is destined to lose the game anyway is less constrained; he might try very strange moves to prolong the play.

Although most non-canonical moves lead to quick defeats, the number of possibilities is substantial and we have not attempted a complete search or further arguments that might greatly reduce the size of the requisite search.

Mathematical Go includes a collection of powerful techniques for rigorous analysis of intricate endgame positions, including the four problems at the beginning of this paper. The techniques of Mathematical Go provide a "canonical" line of play, which leads to a best-possible score. If any strategy can win, then the canonical strategy will win. In the problems considered here, a search of only five lines of play, summarized in Table 2, was sufficient to determine that White wins Problems 3 and 4. The same analysis shows that the outcome of Problems 1 and 2 depends on a Chinese kofight at $\mathrm{S}^{RLLLLLR}$. This location, not E, is the Thousand-Dollar Ko. The mathematics proves that, against a determined opponent, neither player can win Problem 1 or Problem 2 without creating this ko and then winning it.

However, the mathematical techniques used in our analysis of hot and tepid games do not say anything about the number of pairs of dame remaining, nor about the number of moves required to conclude the game. Since the outcome of the Chinese kofight depends on the number of dame, we are not even completely sure who can win! It is conceivable that still other non-canonical lines of play not yet discovered might leave more or fewer pairs of dame on the board when the crucial kofight at $S^{RLLLLLR}$ begins. But based on everything we know, we now think that

- White wins Problem 1 by one point after 95 moves.
- Black wins Problem 2 by one point after 90 moves.
- White wins Problem 3 by one point after 79 moves.
- White wins Problem 4 by one point after 77 moves.

Readers who yearn for another problem of this type will enjoy Figure C.19 of *Mathematical Go*, which White wins in 101 moves. The software companion contains a degenerate version of this problem, just as Problems 3 and 4 above are degenerate versions of Problems 1 and 2.

References

[Berlekamp and Wolfe 1994] E. Berlekamp and D. Wolfe, *Mathematical Go: Chilling Gets the Last Point*, A. K. Peters, Wellesley, MA, 1994. Also published in paperback, with accompanying software, as *Mathematical Go: Nightmares for the Professional Go Player*, by Ishi Press International, San Jose, CA.

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